

A TRANSFER FUNCTION MODEL FOR PROPAGATION IN HOMOGENEOUS MEDIA

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Abstract

Diffraction effects are important in acoustic imaging and tissue characterization because of the relatively large wavelengths used and the fact that applications are frequently used in the near-field of the source. It is difficult to intuitively anticipate the shape of the field there, yet the description of the field's spatial acoustic potential or pressure distribution is necessary. This problem is more complicated when focused transducers or phased arrays are used. Using the spatial frequency, domain it is possible to model propagation in lossless and lossy media as a transfer function. The sources are represented as planar sources with separable arbitrary time excitation and arbitrary spatial excitation. Transfer functions can be obtained for lossless media, media with a linear frequency dependence of attenuation coefficient, and media with a quadratic dependence of attenuation coefficient. The transfer functions are shown to be simply related to the two-dimensional spatial transform of the Green's function of the wave equation for propagation in the medium of interest with the assumed boundary conditions. The transfer functions of the lossy and lossless propagation models are shown to be interdependent. For any given observation plane, these transfer functions are time-varying spatial filters that attenuate higher spatial frequencies with increasing effectiveness as time proceeds. The effects of source excitation and apodization, source boundary conditions, assumed media properties, and receiver aperture effects are easily incorporated in this model. Several numerical simulations of computed acoustic potentials and pressure distributions are shown.

Introduction

The design of pulsed focussed transducers, transducer arrays, and apodized transducers requires an efficient mathematical technique to rapidly compute the acoustic fields (either acoustic potential or acoustic pressure) to evaluate the effects of the design on peak pressure levels, focusing resolution, and other parameters of interest to the designer. Given a source of arbitrary shape, and temporal and spatial excitation, we seek a method to compute the time-varying fields in front of the transducer. Current mathematical techniques¹⁻⁸ involve evaluation of complicated line integrals, and require different integrals depending on the observation point location. Additionally, most techniques are limited to application in lossless media, while computation of the fields is frequently required in lossy media such as absorbent liquids or human tissue. This paper presents an efficient computer-based technique that allows computation of the acoustic potential or pressure in an observation plane in various media, both lossless and lossy. This technique should be of interest to those doing computer-aided ultrasonic transducer design for medical and other applications.

The problem is shown in Fig. 1. Given a planar source with arbitrary (but separable) temporal and spatial excitation, we seek to calculate the acoustic potential $Q(x,y,z,t)$ at a plane located a distance z from the source. (A weakly-focussed transducer or array can also be modeled as a planar source that allows use of this method.) We will assume that the normal velocity, $v_z(x,y,0,t)$, is of the separable form

$$v_z(x,y,0,t) = T(t)s(x,y) \quad (1)$$

where $s(x,y)$ is the arbitrary spatial excitation and $T(t)$ is an arbitrary temporal excitation.

Theory

Since propagation in linear homogeneous media is a linear time-invariant process, we are able to use linear systems concepts to solve the problem under consideration. In this section we define some terms and review results predicted by linear systems concepts.

As shown in Figure 2, the impulse response $g(x,y,z,t)$ is the potential that results from a source of the form $\delta(x-x_s, y-y_s, 0, 0)\delta(t)$ located at a position (x_s, y_s) in the source plane. The form of $g(x,y,z,t)$ will depend on the wave equation of the medium and on the boundary conditions of the source plane. The function $g(x,y,z,t)$ is commonly called the "Green's function" when it is found from the wave equation and boundary conditions. Since the other simple boundary conditions have been shown to be easily expressed in terms of a rigid baffle at the source, we will assume a rigid baffle for the boundary conditions of the source of the paper.

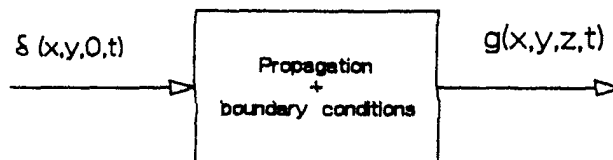
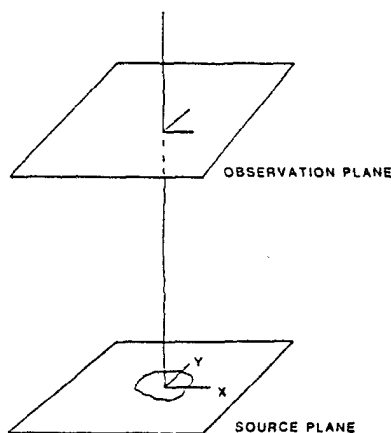


Figure 1 Geometry of propagation problem

Figure 2 Block diagram of impulse response

As shown in Fig. 3, the spatial impulse response of the medium is defined^{1,2} as the response to a source of the form $s(x,y)\delta(t)$. For lossless and linear media, the spatial impulse response is given^{10,11} as

$$h(x,y,z,t) = s(x,y) \ast g(x,y,z,t) \quad (2)$$

where $g(x,y,z,t)$ is the Green's function of the wave equation. For the quadratic medium, the corresponding relationship has been shown¹² to be,

$$h(x,y,z,t) = s(x,y) \ast \left[g(x,y,z,t) + \beta \frac{\partial g}{\partial t} \right] \quad (3)$$

where β is a loss term defined later in this paper.

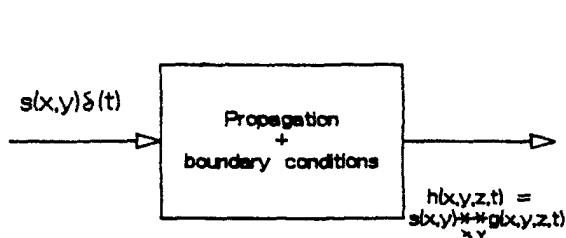


Figure 3 Block diagram of spatial impulse response

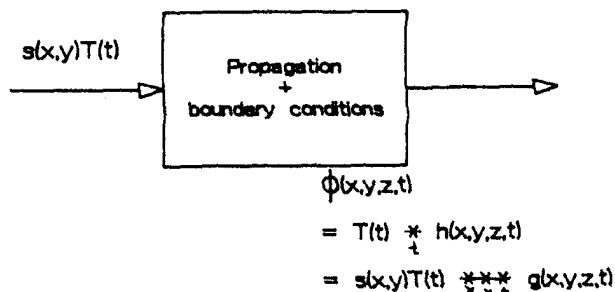


Figure 4 Block diagram of general solution

The double spatial convolutions indicated are simpler multiplication in the Fourier domain. Denoting the two-dimensional spatial transform by a tilde, Eq. 3 becomes

$$\tilde{h} = \tilde{s} \left[\tilde{g} + \beta \frac{\partial \tilde{g}}{\partial t} \right] \approx \tilde{s} \tilde{g} \quad (4)$$

where the last term of Eq. 4 has been neglected since $\beta = 10^{-10}$ for most liquids and gases. This last approximation produces the same result as spatially transforming Eq. 2.

As shown in Fig. 4, the acoustic potential $\phi(x, y, z, t)$ can be found from the spatial impulse response or from the Green's function. The equation is

$$\phi(x, y, z, t) = T(t) * h(x, y, z, t) = T(t) s(x, y) *_{x, y} g(x, y, z, t) \quad (5)$$

In our work we have done the temporal convolution directly, while performing the spatial convolutions as multiplications in the spatial frequency domain. The equation for this is

$$\phi(x, y, z, t) = T(t) * \mathcal{F}^{-1}\{\tilde{s}\tilde{g}\} \quad (6)$$

This last equation expresses the method used out technique. Given the input velocity, $v_z = s(x, y)T(t)$, we

- 1) find the two-dimensional transform $\tilde{S}(f_x, f_y)$ of the spatial portion $s(x, y)$,
- 2) multiply $\tilde{S}(f_x, f_y)$ by the appropriate transfer function $\tilde{g}(f_x, f_y, z, t)$ for the propagation under consideration with the boundary conditions specified,
- 3) perform the inverse two-dimensional spatial transform to find the spatial impulse response $h(x, y, z, t)$,
- 4) perform a temporal convolution of $h(x, y, z, t)$ with the temporal portion of the input excitation $T(t)$ to find the desired acoustic potential $\phi(x, y, z, t)$, and
- 5) compute the pressure $p(x, y, z, t)$, if desired, from the acoustic potential with the relation.

The remainder of this paper presents some models for the propagation and results based on those models.

Wave equations and Green's functions

Three wave equations given in Eqs. 7-9 can be used to model propagation in lossless media, in media with a loss coefficient that is linear in frequency over the frequency range from 1 to 10 MHz (called "linear-loss" media, hereafter), and in media with a loss coefficient that varies quadratically with frequency over a portion of its frequency range (called "quadratic-loss" media, hereafter).

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0 \quad (\text{lossless media}) \quad (7)$$

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - A \frac{\partial \phi}{\partial t} + B \phi = 0 \quad (\text{linear loss}) \quad (8)$$

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \beta \frac{\partial \nabla^2 \phi}{\partial t} = 0 \quad (\text{quadratic loss}) \quad (9)$$

The first equation is the standard wave equation. The second equation is the "Irvine" model presented in Ref. 13. (Note that if $B = 0$, then the equation is the "telegrapher's equation" that was proposed earlier¹⁴ for this medium.) This equation has a plane-wave attenuation coefficient that is linearly dependent on frequency over a portion of its frequency response. The third equation is the Stoke's equation and is presented in standard acoustics texts as representing propagation in media with a quadratic frequency dependence of the plane-wave attenuation coefficient. (Note in Eqs. 8 and 9 that if the loss terms A , B , and β approach zero that the wave equations will revert to the lossless case. This behavior was used as one of the tests to check the validity of the solutions obtained.)

The Green's functions for these equations are known^{13,12} and written in Eqs. 10-12.

$$g_1(x, y, z, t) = \frac{\delta(ct - R)}{2\pi R} \quad (\text{lossless media}) \quad (10)$$

$$g_2(x, y, z, t) = e^{-Ac^2t} \left[\frac{\delta(ct - R)}{2\pi R} + \frac{\sqrt{B + (A^2c^2/4)} I_1 \left[\sqrt{B + (A^2c^2/4)} \sqrt{c^2t^2 - R^2} \right]}{2\pi \sqrt{c^2t^2 - R^2}} \right] \quad (\text{linear loss}) \quad (11)$$

$$g_3(x, y, z, t) \approx \frac{2 \exp \left[- \left(ct - \sqrt{r^2 + c^2t^2} \right)^2 / 2c^2\beta t \right]}{c\sqrt{\beta t} \sqrt{r^2 + z^2}} \quad (\text{for } \beta\omega^2t \leq 100) \quad (\text{quadratic loss}) \quad (12)$$

The transfer functions for propagation in the media of interest for a rigid baffle are written in Eqs. 13-15.

$$\tilde{g}_{p1} = J_0 \left(\rho \sqrt{c^2t^2 - z^2} \right) H(ct - z) \quad (\text{lossless media}) \quad (13)$$

$$\tilde{g}_{p2} = I_0 \left[\sqrt{\rho^2 - B - (A^2c^2/4)} \sqrt{c^2t^2 - z^2} / 2 \right] \exp(-Ac^2t/2) H(ct - z) \quad (\text{linear loss}) \quad (14)$$

$$\tilde{g}_{p3} \approx \exp(-2\pi^2c^2\beta\rho^2t) \left[\frac{\exp(-z^2/2c^2\beta t)}{c\sqrt{\beta t}} ; J_0 \left(2\pi \sqrt{c^2t^2 - z^2} \right) H(ct - z) \right] \quad (\text{for } \beta\omega^2t \leq 100) \quad (\text{quadratic loss}). \quad (15)$$

Numerical simulations

The following simulations have been done using a 64x64 array of data for 50 points in time. While the method gives a three-dimensional solution at any given observation distance, one dimension is eliminated in the plots by representing the solution through a median of the source, as is conventionally done in the literature. The plots show the amplitude of the wave plotted against cross-direction and time. For plotting convenience, the plots have been normalized to the maximum amplitude value obtained for lossless propagation. The width is normalized to the characteristic source size D (i.e., either the diameter or the width), and the time axis is normalized by the value of D/c . The origin of the time axis begins at z/c , the instant that the first part of the wave arrives at the observation plane. All plots are in an observation plane located 10 cm in front of the source plane.

Figures 5-7 show the calculated impulse response from a square piston source (i.e., $s(x, y)$ is a uniform square). The values of the loss coefficient in the lossy media are given in the captions. The lossy media are seen to attenuate the waves and to cause a filling in of the region between the "tails" of the wave as time proceeds. Also the lower spatial frequencies are seen to increasingly dominate as time increases.

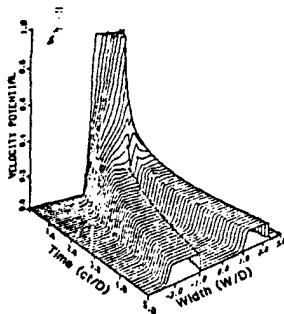


Figure 5. Square transducer, impulse excitation, $z=10$ cm, lossless diffraction, $D=2.2$ cm

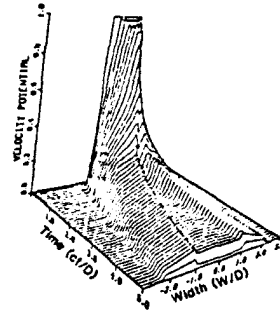


Figure 6. Square transducer, impulse excitation, $z=10$ cm, lossy medium ($B+(A^2c^2/4)=1.266$), $D=2.2$ cm

To illustrate a spatially nonuniform excitation, we consider a circular region (diameter is D) with a Gaussian spatial excitation. The $1/e$ widths are indicated in the captions. The calculated impulse responses are shown in Figs. 8-10. The shape of the Gaussian wave stays much the same in both the low-loss and high-loss cases because of the lower spatial frequency content of this waveshape.

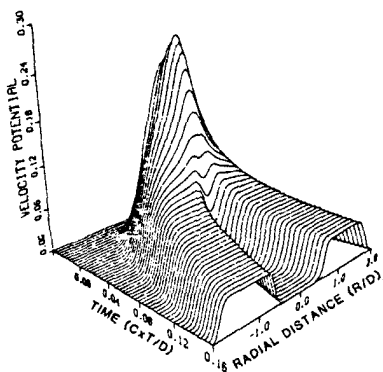


Figure 7. Square transducer, impulse excitation, $z=10$ cm, lossy medium ($\beta = 10^{-9}$ s), $D=3.1$ cm

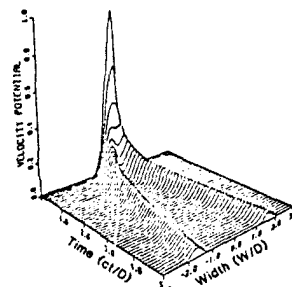


Figure 8. Circular source, Gaussian spatial excitation, $z=10$ cm, lossless medium, $1/e$ point=0.491 cm, $D=2.2$ cm

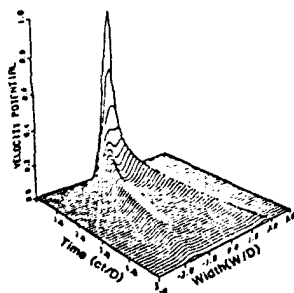


Figure 9. Circular source, Gaussian spatial excitation, $z=10$ cm, lossy medium ($B+(A^2 c^2/4)=1.266$) $1/e$ point=0.491 cm, $D=2.2$ cm

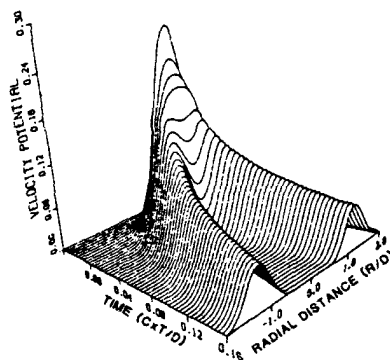


Figure 10. Circular source, Gaussian spatial excitation, $z=10$ cm, lossy medium ($\beta = 10^{-9}$ s), $1/e$ point = 1.18 cm, $D=3.1$ cm

For a time excitation different than $\delta(t)$, the diffracted wave is a convolution between the impulse response and the time derivative of the temporal excitation portion of the source acoustic potential given by Eq. 1. The pressure of the wave is proportional to the time derivative of the acoustic velocity potential. Figure 11 shows the pressure pattern from a uniformly excited square velocity with a one-cycle square wave temporal excitation. The period of the square wave is $8 \times 10^{-3} D/c$. The effects of the time derivative are noticeable along the time axis.

Summary

This paper presents a computationally efficient method of computing the transient acoustic waves in lossless and lossy media. The fields are expressed in terms of the spatial impulse response which is found by inverse transforming the product of the transform of the spatial excitation and the appropriate propagation transfer function for the medium. No geometrical interpretations are required as the method uses only the spatial Fourier transform (or Hankel transform) in its computations.

Acknowledgments

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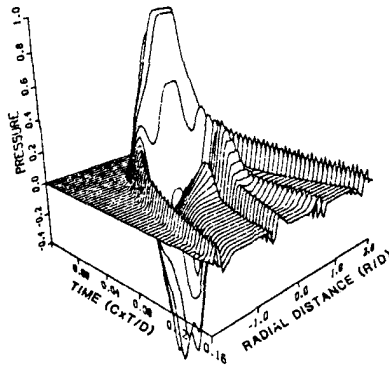


Figure 11. Transient pressure response from square transducer with a single-cycle square wave excitation with a period of 8×10^{-5} D/c seconds ($z=10$ cm, $D=3.1$ cm, $\beta=10^{-10}$ s)

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